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Vortex strings and four-dimensional gauge dynamics

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Abstract: We study the low-energy quantum dynamics of vortex strings in the Higgs phase of $\mathcal{N} = 2$ supersymmetric QCD. The exact BPS spectrum of the stretched string is shown to coincide with the BPS spectrum of the four-dimensional parent gauge theory. Perturbative string excitations correspond to bound W-bosons and quarks while the monopoles appear as kinks on the vortex string. This provides a physical explanation for an observation by N. Dorey relating the quantum spectra of theories in two and four dimensions.

Keywords: Nonperturbative Effects, Solitons Monopoles and Instantons, Supersymmetric Effective Theories.

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1. Introduction and conclusion

Two-dimensional sigma-models have long acted as a playground in which to test aspects of four-dimensional gauge dynamics. The two systems share many qualitative features including asymptotic freedom, a dynamically generated mass gap, anomalies, and instantons.

Some years ago, N. Dorey proved a more quantitative correspondence between supersymmetric theories in two and four dimensions [1]. He showed that the BPS spectrum of the mass deformed two-dimensional $\mathcal{N} = (2,2)$ $\mathbf{CP}^{N-1}$ sigma-model coincides with the BPS spectrum of four-dimensional $\mathcal{N} = 2$ SU(N) supersymmetric QCD. The correspondence is exact, holding at the quantum level in both weak and strong coupling regimes. Generalisations to other two-dimensional sigma-models were later found [2]. However, despite some insight from brane constructions [3, 2], the underlying reason for the agreement remained mysterious. The purpose of this paper is to provide a field theoretic explanation for the correspondence.

The key to our story lies in the recent progress in understanding the dynamics of various soliton configurations in the Higgs phase of $\mathcal{N} = 2$ SQCD. Of particular relevance are the non-abelian vortices [4], which are string-like objects in four dimensions, and monopoles [5] which, due to the Meissner effect, are confined in the Higgs phase and come attached to two semi-infinite vortex strings.[1] We shall show that the two-dimensional

[1]Analogous configurations in a closely related theory were also discussed in [6] and [7] respectively. Other work on confined monopoles can be found in [8].

theory considered by Dorey in [1] is precisely the theory describing the vortex string. As we explain below, the BPS excitations of the string have an interpretation as four dimensional states: the perturbative string excitations correspond to W boson-string bound states, while the solitonic kinks of the string correspond to the confined monopoles in four dimensions [5].

The results presented here fit into the growing body of work devoted to understanding the dynamics of solitons in the Higgs phase of $\mathcal{N} = 2$ theories. In recent years we have found that these theories admit a remarkably rich structure of classical BPS solitons. As well as the strings and confined monopoles mentioned above, there is an intricate system of domain walls [9, 10], domain wall junctions [11] and, perhaps most remarkably, D-branes [12, 13] in which the vortex string terminates on a domain wall where its end is electrically charged under a gauge field. While field theoretic D-branes have been known to exist for some time in strong coupling regimes [14] the objects described in [12, 13] are amenable to semi-classical analysis.

The correspondence discovered in [1] holds in both strong coupling and weak coupling regimes of the two theories. In the latter regime, the central charge of the theory may be expanded in an infinite series of instanton contributions. Since the BPS spectra coincide, this expansion agrees term by term, suggesting a quantitative correspondence between two dimensional instantons (which are vortices) and four-dimensional Yang-Mills instantons. Indeed, in [4], an ADHM-like construction of the vortex moduli space was presented and it was shown that the moduli space of vortices is a particular submanifold of the moduli space of instantons. It would be interesting to prove explicitly that the integrals over the relevant moduli spaces coincide. From the interpretation of the correspondence presented here, this agreement suggests another solitonic connection: a vortex in a vortex string looks like a Yang-Mills instanton in four dimensions. In section 3, we present the Bogomoln'yi equations describing such a solution.

The conclusion of this paper — that the quantum dynamics of solitons, specifically vortex strings, may be used to extract information about the strong coupling dynamics of the underlying four dimensional gauge theory — is reminiscent of the stringy games played in ten dimensions. For example in the the old-new Matrix theory, D-brane solitons contain much information about the bulk dynamics. It would be interesting to see if this analogy can be pushed further.

The plan of the paper is as follows. In section 2, we study $\mathcal{N} = 2$ supersymmetric $\mathrm{U}(N_c)$ gauge theory with $N_f = N_c$ flavours. We review the exact central charge on the Coulomb branch which can be determined from the Seiberg-Witten solution. We then follow the states as you slide onto the Higgs branch, breaking the gauge group completely. We shall show that the monopoles remain BPS, but are now confined. At the same time, a new BPS object appears: the vortex string. In section 3 we describe the low-energy dynamics of the vortex string and show that it is governed by the two-dimensional theory studied in [1]. We review the computation of the BPS spectrum and confirm that it does indeed coincide with that of the four-dimensional parent theory. In particular, we shall see that elementary excitations of the string are associated to W-bosons, while kinks in the string are monopoles in four dimensions. In section 4, we repeat this story for $N_f > N_c$

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flavours, and the associated "semi-local" vortices, giving a rationale for the generalisation discovered in [2]. This includes the case of the conformal vortex string.

Note added: after finishing this work we were informed that similar conclusions have been reached by M. Shifman and A. Yung [15]. We would like to thank M. Shifman for communicating their results to us prior to publication.

2. The four dimensional gauge theory: $N_f = N_c$

Our interest in this paper will focus on $\mathcal{N} = 2$ supersymmetric QCD with $\mathrm{U}(N_c)$ gauge group with N_f flavours transforming in the fundamental representation. In this section we restrict to $N_f = N_c$. Nevertheless we shall continue to use the subscripts f and c to distinguish between flavour and colour groups. Generalisations to $N_f > N_c$ will be given in section 4. We denote the complexified gauge coupling constant[2] as $\tau = 2\pi i/e^2 + \theta/2\pi$.

In $\mathcal{N} = 1$ language the theory contains a $\mathrm{U}(N_c)$ vector multiplet field, an adjoint chiral multiplet Φ and a further $2N_f$ chiral multiplets Q_i and $\tilde{Q}_i$, $i = 1, \ldots, N_f$. The Q_i transform in the $(\mathbf{N}_c, \bar{\mathbf{N}}_f)$ of the $\mathrm{U}(N_c) \times \mathrm{SU}(N_f)$ gauge and flavour group; the $\tilde{Q}_i$ transform in the $(\bar{\mathbf{N}}_c, \mathbf{N}_f)$. The lowest component of each chiral multiplet is a complex scalar field which, as is traditional, we denote by the corresponding lower-case letter i.e. ϕ, q_i and $\tilde{q}_i$. We provide each of the hypermultiplets with a complex mass parameter m_i through the superpotential,

$$\mathcal{W} = \sqrt{2} \sum_{i=1}^{N_f} \tilde{Q}_i (\Phi - m_i) Q_i \,.$$

Generically the masses break the flavour group of the theory $\mathrm{SU}(N_f) \xrightarrow{m} \mathrm{U}(1)^{N_f - 1}$. The lagrangian also enjoys an $\mathrm{SU}(2)_R \times \mathrm{U}(1)_R$ classical R-symmetry. In the presence of non-zero masses, the latter is broken to Z_2.

The theory has an intricate moduli space of vacua depending on the hypermultiplet masses m_i, as well as a Fayet-Illiopoulos (FI) parameter which we shall introduce shortly. For now, we take this FI parameter to vanish, ensuring that there is always a Coulomb branch of vacua parameterised by $\phi = \mathrm{diag}(\phi_1, \ldots, \phi_{N_c})$ in which the gauge group is generically broken to the Cartan subalgebra $\mathrm{U}(N_c) \xrightarrow{\phi} \mathrm{U}(1)^{N_c}$. When some of the masses coincide, one can also have Higgs branches of vacua parameterised by holomorphic gauge invariant operators formed from the hypermultiplet fields. For $N_f = N_c$, these include the baryonic operators,

$$\begin{aligned} B &= Q_1^{a_1} Q_2^{a_2} \cdots Q_{N_c}^{a_{N_c}} \, \epsilon_{a_1 \ldots a_{N_c}} \\ \tilde{B} &= \tilde{Q}_1^{a_1} \tilde{Q}_2^{a_2} \cdots \tilde{Q}_{N_c}^{a_{N_c}} \, \epsilon_{a_1 \ldots a_{N_c}} \end{aligned}$$

where a_i denote colour indices. There are also meson operators of the form $M_{ij} = \tilde{Q}_i Q_j$. The pattern in which the vacuum moduli spaces intertwine was described in detail in [16]

[2] A note on conventions: our Yang-Mills term is normalised as $(1/4e^2)\,\mathrm{Tr}(F_{\mu\nu}F^{\mu\nu})$ which differs by a quadratic Casimir factor of 2 from the usual conventions. This leads to an unfamiliar factor of 2 in this and other formulae containing e^2.

for the $\mathcal{N}=2$ theories of interest. (The surviving vacua for $\mathcal{N}=1$ deformations were examined in [17]).

The classical spectrum of BPS states depends on the vacuum in which the theory lives. We shall start by discussing the classical spectrum on the Coulomb branch, only subsequently moving onto quantum corrected spectrum and, ultimately, to the quantum spectrum on the Higgs branch. At a generic point on the Coulomb branch the theory has an interesting mixture of BPS states arising from both elementary excitations as well as non-perturbative monopole and dyon states. Among the former are the N_c massless photons, together with $N_c(N_c-1)$ W-bosons with mass $|\phi_a-\phi_b|$ for $a,b=1,\dots,N_c$. There are also N_cN_f BPS quark states which, for $a=1,\dots,N_c$ and $i=1,\dots,N_f$ have masses given by,

$$M_{\text{quark}}=|\phi_a-m_i|\,. \tag{2.1}$$

All further BPS states arise as solitons and have non-zero magnetic charges under the unbroken gauge group $\mathrm{U}(1)^{N_c}$. We denote these magnetic charges as h_a and require $\sum_a h_a=0$, reflecting the fact that monopole solutions only exist in the semi-simple $\mathrm{SU}(N)_C\subset\mathrm{U}(N)_C$ part of the gauge group. The classical mass of these monopoles is given by

$$M_{\text{mon}}=\frac{2\pi}{e^2}\left|\sum_{a=1}^{N_c}h_a\phi_a\right|. \tag{2.2}$$

In addition to these purely magnetic solitons, the classical spectrum also contains an infinite tower of dyons. A unified mass formula for each of these objects can be given in terms of the central charge Z. For BPS states with electric charge j_a and magnetic charge h_a under $\mathrm{U}(1)^{N_c}$, and with charge s_i under the global flavour group $\mathrm{U}(1)^{N_f-1}$, the mass of any BPS state is given by $M=|Z|$ with

$$Z=\sum_{a=1}^{N_c}\phi_a(j_a+\tau h_a)+\sum_{i=1}^{N_f}m_is_i\,. \tag{2.3}$$

The above discussion has been classical. Let us now turn to various aspects of the quantum theory. The overall U(1) part of the gauge group becomes weakly coupled in the infrared[3] and the interesting dynamics lies in the interactions of the $\mathrm{SU}(N)$ part of the gauge group. For vanishing $m=\phi=0$, the one-loop beta-function for the $\mathrm{SU}(N_c)$ gauge coupling has a coefficient proportional to $-(2N_c-N_f)=-N_c$ and the gauge coupling e^2 runs logarithmically with the scale μ. It can be eliminated in favour of an RG invariant scale,

$$\Lambda=\mu\exp\left(-\frac{4\pi^2}{N_c\,e^2(\mu)}\right). \tag{2.4}$$

Another quantum effect which will be important in the following arises from anomalies: the $\mathrm{U}(1)_R$ symmetry is broken by instantons to $\mathbf{Z}_{2(2N_c-N_f)}=\mathbf{Z}_{2N_c}$ when $m_i=0$. (Recall that, in the presence of hypermultiplet masses m_i, $\mathrm{U}(1)_R$ is further broken at the classical level to Z_2).

[3] Readers uncomfortable with the Landau pole are free to turn on a noncommutivity parameter and repeat the story below.

Most important for our purposes are the quantum corrections to the masses of BPS states. At weak coupling $|\phi_a - \phi_b| \gg \Lambda$, one can show that the mass formula receives contributions from one-loop effects, together with an infinite series of instanton corrections. At strong coupling one needs another technique to compute the spectrum. Thankfully a beautiful method is provided by Seiberg and Witten's famous solution to the low-energy dynamics on the Coulomb branch [18]. We now review the Seiberg-Witten solution for the exact central charge Z evaluated at a specific point on the Coulomb branch.

2.1 At the root of the baryonic Higgs branch

For reasons that will shortly become clear, we will be interested in the BPS spectrum of the theory arising at a point on the Coulomb branch known as the "root of the baryonic Higgs branch"[4] [16]. This is the point defined classically by $\phi = \mathrm{diag}(m_1, \ldots, m_{N_c})$. so that the breaking of flavour and gauge symmetries occurs at the same scale $\mathrm{U}(N_c) \times \mathrm{SU}(N_f) \xrightarrow{m} \mathrm{U}(1)^{N_c} \times \mathrm{U}(1)^{N_f-1}$. From equation (2.1) we see that N_c of the $N_f N_c$ degrees of quark freedom become massless at this point. In fact, the quark masses become precisely degenerate with the masses of photons and W-bosons, each of which have classical masses for given by

$$M_{\mathrm{W-boson}} = M_{\mathrm{quark}} = |m_i - m_j|\,. \tag{2.5}$$

Because of this degeneracy the classical central charge (2.3) may be written in the simplified form,

$$Z = \sum_{i=1}^{N_c} m_i (S_i + \tau h_i) \tag{2.6}$$

where we have redefined the charges as $S_i = s_a + j_a$. We would now like to describe the quantum corrections to this charge formula as encoded in the Seiberg-Witten solution. (Recently the semi-classical computation of corrections to the monopole mass was revisited in [13, 19], finding agreement with the exact result of Seiberg and Witten). At the root of the baryonic Higgs branch, the Seiberg-Witten elliptic curve has a special property: it degenerates [16]

$$F(t,u) = \left(t - \prod_{i=1}^{N_c}(u - m_i)\right)\left(u - \Lambda^{N_c}\right). \tag{2.7}$$

This form of the curve occurs naturally in the M-theory construction of [20], where the degeneration corresponds to the fact that one of the IIA NS5 branes remains unbent upon its ascent to M-theory. The curve is branched over the N_c points e_i defined by,

$$\prod_{i=1}^{N_c}(u - m_i) - \Lambda^{N_c} = \prod_{i=1}^{N_c}(u - e_i) = 0\,. \tag{2.8}$$

[4] In the present context, with $N_f = N_C$, there is no Higgs branch emanating from this point even when $m_i = 0$. A better name might be "root of the baryonic Higgs phase".

In the quantum theory the central charge is given by the integral of the Seiberg-Witten differential $\lambda_{SW} = (u/t)dt$ over certain one cycles of the curve. The resulting modification of the classical formula (2.6) is

$$Z = \sum_{i=1}^{N_c} (m_i S_i + m_{Di} h_i) \tag{2.9}$$

where all the quantum corrections are encoded in the functions $m_{D\,i}$ which are holomorphic in the hypermultiplet masses m_i and Λ. They are given by

$$m_{Dl} - m_{Dk} = \frac{1}{2\pi i}\int_{e_k}^{e_l} d\lambda_{\rm SW} = \frac{1}{2\pi i}\int_{e_k}^{e_l} u\frac{dt}{t} = \frac{1}{2\pi}\sum_{i=1}^{N_c}\int_{e_k}^{e_l}\frac{u\,du}{u-m_i}$$

where, in the final equality, we have used the exact form of the curve (2.7). Evaluating this integral, we find the expression for the contribution to the central charge given by

$$m_{Dl} - m_{Dk} = \frac{1}{2\pi}N_c(e_l - e_k) + \frac{1}{2\pi}\sum_{i=1}^{N_c} m_i \log\left(\frac{e_l - m_i}{e_k - m_i}\right). \tag{2.10}$$

2.2 On the baryonic Higgs branch

The Seiberg-Witten computation of the spectrum holds on the Coulomb branch and we have presented the result above at a very specific point, known as the root of the baryonic Higgs branch. Let us now ask what becomes of the BPS spectrum as we move *onto* the baryonic Higgs branch. We do this by turning on a Fayet-Illiopoulos (FI) parameter v^2 for the U(1) part of the gauge theory, so the D-term becomes,

$$D = \sum_{i=1}^{N_f} q_i q_i^\dagger - \tilde{q}_i^\dagger \tilde{q}_i - v^2\,.$$

The FI parameter v^2 lifts the Coulomb branch and forces the theory onto the Higgs branch. The theory has a unique vacuum state, given by

$$\phi = {\rm diag}(m_1, \ldots, m_{N_c})\,, \qquad B = v^{N_f}\,, \qquad \tilde{B} = M = 0 \tag{2.11}$$

We now see why the point $\phi = {\rm diag}(m_1, \ldots, m_{N_c})$ is called the root of the baryonic Higgs branch: it indeed provides the gateway into the Higgs phase when the FI parameter is turned on. The pattern of symmetry breaking in this vacuum is given by

$$\mathrm{U}(N_c) \times \mathrm{SU}(N_f) \xrightarrow{m} \mathrm{U}(1)^{N_c} \times \mathrm{U}(1)^{N_f - 1} \xrightarrow{v} \mathrm{U}(1)^{N_c-1}_{\rm diag} \tag{2.12}$$

Our interest remains on the spectrum of BPS states, but now in the vacuum (2.11). What becomes of the various BPS states as we turn on the FI parameter v^2? Let us firstly consider elementary excitations. The photons and W-bosons pick up an extra contribution to their mass proportional to ev through the Higgs mechanism. In doing so, they combine with the $N_f N_c = N_c^2$ quark hypermultiplets and are no longer BPS, now sitting in long supersymmetry multiplets[5] [23]. None of the elementary particle states remain BPS.

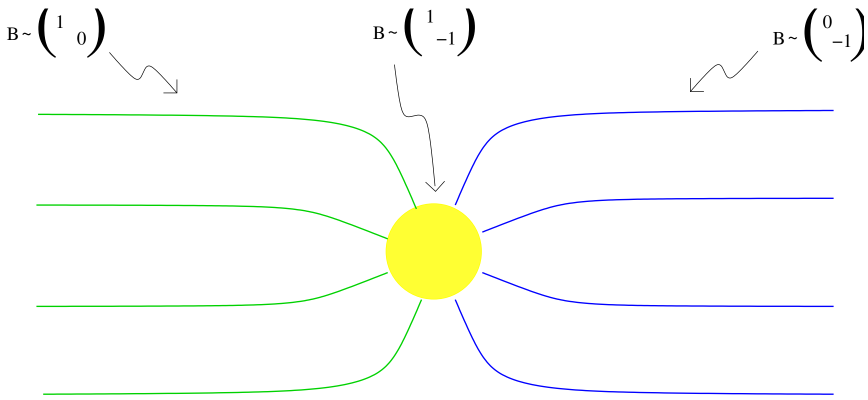

Figure 1: A sketch of the U(2) monopole in the Higgs phase [5]. The caricature is accurate for $ev \ll |m_i - m_j|$, a limit which ensures the flux tube is larger than the monopole core.

Let us now turn to the magnetic monopoles. At first sight, it appears unlikely that they can remain BPS on the Higgs branch. Since the gauge group is fully broken, the Meissner effect ensures that magnetic flux can no longer freely permeate the vacuum but is restricted to lie in a flux tube. Thus the monopoles are confined and, in isolation, have infinite mass. Nevertheless, as shown in [5], the monopoles are BPS. The final object can be thought of as the original monopole, now emitting two vortex strings and the total combination preserves 1/4 of the original supersymmetry. The classical Bogomoln'yi equations describing this monopole-flux-tube combo can be derived by completing the square in the hamiltonian. Denoting the non-abelian magnetic field as B and setting all irrelevant fields to zero we choose that the monopole ejects its flux tubes in the x^3 directions. We manipulate the hamiltonian thus [5]

$$\begin{aligned}
\mathcal{H} &= \frac{1}{2e^2}B^2 + \frac{1}{2e^2}|\mathcal{D}\phi|^2 + |\mathcal{D}q_i|^2 + \frac{e^2}{2}(q_i q_i^\dagger - v^2)^2 + q_i^\dagger|\phi - m_i|^2 q_i \\
&= \frac{1}{2e^2}(\mathcal{D}_1\phi - B_1)^2 + \frac{1}{2e^2}(\mathcal{D}_2\phi - B_2)^2 + (\mathcal{D}_3\phi - B_3 - e^2(q_i q_i^\dagger - v^2))^2 + \\
&\quad + |\mathcal{D}_1 q_i - i\mathcal{D}_2 q_i|^2 + |\mathcal{D}_3 q_i + (\phi - m_i)q_i|^2 + \frac{1}{e^2}\partial_\mu(\phi B_\mu) - v^2 B_3 \\
&\geq \frac{1}{e^2}\partial_\mu(\phi B_\mu) - v^2 B_3
\end{aligned} \tag{2.13}$$

where we have left colour indices and traces implicit and we have summed over the flavour index i. The Bogomoln'yi equations can be found in the total squares on the second line. While no explicit solutions to these equations are known, several properties were deduced in [5].[6] We draw a caricature of the solution in figure 1.

The two terms in the final line of (2.13) measure conserved topological charges. The first is precisely the magnetic charge carried by the monopole. In the Coulomb phase the

[5]This issue also arose in [21, 22] where it was argued that, in certain theories, they remain "almost BPS".

[6]As explained in [13], solutions to these equations describe a wider range of objects than the confined monopoles considered here and include strings ending on domain walls.

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integral $\int d^3x\, \partial\cdot(\phi B)$ is evaluated on the S^2_∞ boundary. In the present case, the monopole flux does not make it to all points on the boundary, but is confined to two flux tubes which stretch in the $\pm x^3$ direction. Correspondingly, the integral should now be evaluated over two planes $\mathbf{R}^2_{\pm\infty}$ at $x^3 = \pm\infty$. The second term in (2.13) is new. When integrated over the $x^1 - x^2$ plane, it measures the tension of the flux tubes emitted by the monopole. These are simply vortices, supported by the overall broken U(1) gauge symmetry in (2.12). They have tension given by $T_{\rm vortex} = 2\pi v^2 k$ for $k \in \mathbf{Z}$ and will be the subject of study in section 3. Since these vortex strings have finite tension and semi-infinite length, the total mass of the configuration is infinite. This reflects the fact that the monopoles are confined. Nonetheless, as we see from (2.13), this infinite mass splits unambiguously into a finite contribution from the monopole and an infinite contribution from the flux-tube. As an alternative way of viewing this, we could say that the monopole mass is the excess over an infinite straight vortex string (suitably regulated with an infra-red cut-off). With the monopole's mass defined in this way, we see that it remains identical to that calculated in the Coulomb phase (2.2).

In summary, on the Higgs branch the quarks and W-bosons combine to form long multiplets, while the monopoles are confined yet remain BPS. Moreover, after subtracting the contribution from the BPS flux tube, we have seen that the classical monopole mass remains unchanged as we turn on the FI parameter v^2 and move onto the Higgs branch. Can we understand this and extend the result to the quantum theory? In fact, there is a simple non-renormalisation theorem that tells us that the central charge Z for particle states cannot receive contributions from the FI parameter v^2 and remains given by (2.9) for BPS states on the Higgs branch. The important observation is the fact that, in $\mathcal{N} = 2$ theories, the central charges are given by the scalar components of background vector multiplets [18]. Any dependence on hypermultiplets or linear multiplets (also known as tensor multiplets) is forbidden by supersymmetry. The FI parameter v^2 lies in a background linear multiplet (it is actually one component of a triplet of FI parameters which is precisely the scalar field content of the $\mathcal{N} = 2$ linear multiplet). We therefore conclude that the BPS particle states receive no contribution to their mass from v^2 and the exact quantum corrected central charge on the Higgs branch is given by (2.9).

3. The vortex theory

In the previous section we have derived the BPS spectrum on the baryonic Higgs branch. We have seen that there are no W-boson or quark BPS states, but monopoles survive as BPS objects, albeit confined by the Meissner effect. Moreover, we have something new: a BPS vortex string with tension $2\pi v^2$. In this section, we study the quantum dynamics of this vortex string and show that its mass spectrum reproduces the four dimensional BPS spectrum described above.

Let us start by describing the vortex in the theory with vanishing quark masses, $m_i = 0$. In this case, the lagrangian preserves the full SU(N_f) flavour symmetry but the unique vacuum state on the Higgs branch lies in a colour-flavour locked phase with the symmetry breaking pattern $\mathrm{U}(N_c) \times \mathrm{SU}(N_f) \xrightarrow{v} \mathrm{SU}(N_c)_{\rm diag}$. The breaking of the overall U(1) gauge group ensures that vortex strings are supported with topological winding number given by $\int \mathrm{Tr}\, B = 2\pi k$, with $k \in \mathbf{Z}$, where the integral is taken over the plane transverse to the string. If we choose the strings to lie in the x^3 direction, then the classical configurations obey the non-abelian version of the first order vortex equations

$$B_3 = e^2 \left(\sum_{i=1}^{N_f} q_i q_i^\dagger - v^2 \right) , \qquad \mathcal{D}_1 q_i = i \mathcal{D}_2 q_i \, . \tag{3.1}$$

The strings are BPS with tension given by $T_{\rm vortex} = 2\pi v^2 k$. In [4], an ADHM-like construction of the k-vortex moduli space was derived from a D-brane picture. We review this in appendix A. In the remainder of this paper we shall content ourselves with studying a single vortex $k = 1$. In this case, all zero modes of the vortex are Goldstone modes and the moduli space can be constructed simply from the symmetries of the field theory [4, 6]. The key point to note is that a single non-abelian vortex is simply an abelian Nielsen-Olesen vortex embedded into a U(1) subgroup of U(N_c). Suppose we choose to embed it in the upper left-hand corner. Then acting on this solution with the SU$(N_c)_{\rm diag}$ vacuum symmetry sweeps out a moduli space $\mathcal{M}_{\rm vortex} \cong \mathrm{SU}(N_c)/(\mathrm{U}(1) \times \mathrm{SU}(N_c - 1)) \cong \mathbf{CP}^{N_c - 1}$ of solutions. We therefore have,

$$\mathcal{M}_{\rm vortex} = \mathbf{C} \times \mathbf{CP}^{N_c - 1} \tag{3.2}$$

where $\mathbf{C}$ parameterises the center of mass of the vortex string in the $x^1 - x^2$ plane, while $\mathbf{CP}^{N_c-1}$ describes the internal degrees of freedom arising from the SU$(N_c)_{\rm diag}$ action. The low-energy dynamics of the vortex string can be described by a $d = 1 + 1$ dimensional sigma-model with target space $\mathcal{M}_{\rm vortex}$. Since the vortex is BPS, the low-energy dynamics preserves $\mathcal{N} = (2, 2)$ supersymmetry.

Let us ask how this situation changes for non-zero quark masses m_i. The answer was given in [5]. The masses break the symmetry group as SU$(N_c)_{\rm diag} \rightarrow \mathrm{U}(1)_{\rm diag}^{N_c-1}$, lifting the $\mathbf{CP}^{N_c-1}$ moduli space. For a vortex of unit winding number, there are now N_c isolated solutions corresponding to an abelian vortex embedded in one of the *diagonal* $\mathrm{U}(1) \in \mathrm{U}(N_c)$ subgroups. In other words, the off-diagonal embeddings have been removed. From the perspective of the low-energy dynamics, the masses m_i induce a potential V on $\mathbf{CP}^{N_c-1}$ with N_c isolated minima. This potential is of the form $V \sim K^2$ where K is a holomorphic Killing vector on $\mathbf{CP}^{N_c-1}$. We derive this potential in appendix B.

We now describe the theory in more detail and flesh out some of these results. Firstly, note that our low-energy approach to determine the spectrum of the string is a priori trustworthy provided the string is sufficiently massive: $ev \gg |m_i - m_j|$. In fact, because of the BPS nature of our results, they can ultimately be continued throughout parameter space. With this in mind, we now describe the theory of the vortex. We use the language of the gauged-linear sigma model. This description arises naturally in the brane picture of [4] which we review in appendix A.

Vortex theory: $d = 1 + 1$, $\mathcal{N} = (2, 2)$ *supersymmetric* U(1) *with a single neutral chiral multiplet* Z *and* N_c *chiral multiplets* Ψ_i *of charge* $+1$. *Each charged chiral has twisted* *masses m_i, $i = 1, \ldots, N_c$. The classical theory has dimensionless FI parameter r and vacuum angle θ which are combined in a single complex coupling $\tau = ir + \theta/2\pi$. The gauge theory also contains a dimensionful gauge coupling g.*

A couple of comments are in order. Firstly, the twisted mass in two dimensional gauge theories was introduced in [3]. Each twisted mass is a complex mass for a chiral multiplet, consistent with supersymmetry and gauge invariance. It is forbidden in four-dimensional $\mathcal{N} = 1$ theories by Lorentz symmetry, but becomes available upon dimensional reduction to two dimensions. As our notation suggests, the twisted masses of the vortex theory are identified with the hypermultiplet masses m_i in four-dimensions. This follows immediately from the brane picture of [3] and [4]. The FI parameter of the vortex theory, which determines the Kähler class of the $\mathbf{CP}^{N_c-1}$ moduli space can also be extracted from the brane construction [4]

$$r = \frac{2\pi}{e^2} \,. \tag{3.3}$$

Note that, with this result, the complexified coupling τ of the vortex theory is identified with the complexified coupling τ of the four-dimensional theory.[7] Finally, we are instructed in [4] to take the two-dimensional gauge coupling $g^2 \to \infty$. This arises as a consequence of the decoupling limit of the D-brane system and forces the vortex theory onto its Higgs branch. In what follows, we will leave g^2 finite. This is justified by the existence of the CFIV supersymmetric index [24] which ensures that the BPS spectrum of the vortex theory is independent of g^2.

The neutral chiral multiplet Z contains a single complex scalar field z, parameterising the center of mass motion of the vortex. It corresponds to the $\mathbf{C}$ factor in (3.2). Since this field is free, we pay it no more attention and ignore it in the following. Each charged chiral multiplet Ψ_i also contains a complex scalar ψ_i, $i = 1, \ldots, N_c$, while the U(1) vector multiplet contains the two dimensional gauge field and a further, neutral, complex scalar σ. The bosonic part of the lagrangian describing the internal degrees of freedom of the vortex is given by,

$$-\mathcal{L}_{\text{vortex}} = \frac{1}{2g^2}\left(F_{01}^2 + |\partial\sigma|^2\right) + \sum_{i=1}^{N_c}\left(|\mathcal{D}\psi_i|^2 + |\sigma - m_i|^2|\psi_i|^2\right) + \frac{g^2}{2}\left(\sum_{i=1}^{N_c}|\psi_i|^2 - r\right)^2 \,. \tag{3.4}$$

For vanishing twisted masses m_i, the theory has a $\mathrm{SU}(N_c)_D$ global symmetry which is identified with the $\mathrm{SU}(N_c)_{\text{diag}}$ symmetry in four dimensions. For generic $m_i \neq 0$, this is broken to $\mathrm{U}(1)_D^{N_c-1}$. The theory also has a $\mathrm{U}(1)_R$ symmetry which is inherited from the $\mathrm{U}(1)_R$ symmetry in four dimensions. This rotates the phases of both σ and m_i. For vanishing masses, the vortex theory has a Higgs branch of vacua given by $\sigma = 0$ with the chiral multiplets constrained to obey $\sum_i |\psi|^2 = r$. After dividing by the U(1) action we see the Higgs branch is $\mathbf{CP}^{N_c-1}$ in agreement with (3.2). In the presence of twisted masses,

[7] The identification of the theta angle in four dimensions with the theta angle on the vortex theory is new. It follows simply from the IIA version of the brane construction presented in appendix A. For both the two dimensional theory on the vortex [3] and the four dimensional theory [20], the theta angle is given by the separation of M5-branes along the M-theory circle.

performing the same procedure results in a twisted potential on the Higgs branch of the type constructed in [25] as we show explicitly in appendix B. The potential has N_c isolated vacua given by,

$$\text{Vacuum} \quad i: \qquad \sigma = m_i\,, \qquad |\psi_j|^2 = r\delta_{ij} \tag{3.5}$$

As described above, the i$^{\rm th}$ vacuum corresponds to a vortex embedded in the i$^{\rm th}$ U(1) subgroup, carrying magnetic charge $B = \mathrm{diag}(0,\ldots,0,1,0,\ldots,0)$, where the 1 sits in the i$^{\rm th}$ entry.

So far we have discussed the relevant aspects of the classical two-dimensional theory on the vortex worldsheet. Let us now turn to the quantum theory. When the twisted masses vanish $m_i = 0$, there is a one-loop correction to the FI parameter r, leading to a logarithmic running at scale μ,

$$r(\mu) = r_0 - \frac{N_c}{2\pi}\log\left(\frac{M_{UV}}{\mu}\right) \tag{3.6}$$

where r_0 is the bare FI parameter defined at the UV cut-off M_{UV}. Note that, since this theory describes the low-energy dynamics of a soliton, it is inappropriate to take M_{UV} to infinity. Instead it is set by the mass scale of the vortex: $M_{UV} = v^2$.

In (3.6) we see our first hint that the vortex theory understands something of the four dimensional quantum dynamics since the one-loop beta function for r is identical to that of the four-dimensional coupling e^2. This ensures that the relationship $r = 2\pi/e^2$ is preserved under RG flow. Note that although vortices exist by virtue of the overall $\mathrm{U}(1) \subset \mathrm{U}(N_c)$, the renormalisation of r clearly follows the asymptotically free $\mathrm{SU}(N_c)$ gauge coupling in four dimensions, rather than the infra-red free U(1) coupling. Since the beta functions for r and $2\pi/e^2$ are equal, it follows that if we eliminate $r(\mu)$ in favour of the one-loop RG invariant scale,

$$\Lambda = \mu\exp\left(-\frac{2\pi r(\mu)}{N_c}\right)$$

then this coincides with the dynamically generated scale in four dimensions (2.4).

The anomaly structure provides further agreement between the vortex theory and four dimensions. The $\mathrm{U}(1)_R$ symmetry on the vortex worldsheet is broken by anomalies to $\mathbf{Z}_{2N_c}$, in agreement with the four dimensional result. This suggests an interplay between Yang-Mills instantons and worldsheet instantons. We shall return to this later.

In the presence of twisted masses, the story is similar. The running of the coupling $r(\mu)$ is cut-off at the scale $|m_i - m_j|$. For $|m_i - m_j| \gg \Lambda$, the theory is weakly coupled. Again, this is in agreement with the four dimensional theory at the root of the baryonic Higgs branch, which sits far out on the Coulomb branch when $|m_i - m_j| \gg \Lambda$. In this regime, the N_c classical vacua of the vortex theory (3.5) are trustworthy ground states around which to study excitations. Finally, we note that at strong coupling, $|m_i - m_j| \ll \Lambda$, the Witten index ensures that there remain N_c isolated vacuum states in the quantum vortex theory.

3.1 The spectrum of the vortex string

Having identified the theory on the vortex string and described some of its properties, our task now is to determine its spectrum. In fact this is precisely the calculation performed by Dorey in [1] where he computed the exact quantum BPS spectrum as a function of the twisted masses m_i and Λ. In this subsection we review the results of [1] and describe how they relate to the vortex string.

We deal first with the classical, elementary internal excitations of the BPS string. The vortex theory (3.4) includes a gapped photon with mass $g\sqrt{r}$. This does not lie in a BPS multiplet and, moreover, decouples as $g^2 \to \infty$ so we do not consider it in the following. The elementary BPS excitations arise from the chiral multiplets ψ_i. As we have seen, when the quark masses m_i vanish these parameterise the massless Goldstone modes of the internal $\mathbf{CP}^{N_c-1}$ vortex moduli space of (3.2). In the presence of the masses m_i, these flat directions are lifted and the vortex theory has a classical mass gap. In the $i^{\rm th}$ vacuum, there are $(N_c - 1)$ BPS states arising from the ψ_j with, for $j \neq i$, masses given by

$$M_\psi = |m_j - m_i| \, . \tag{3.7}$$

We see that these perturbative excitations of the string reproduce the classical mass spectrum of the quarks and W-bosons (2.5) in the four dimensional theory, but on the Coulomb branch. Recall that, in the Higgs vacuum we are considering, the classical mass of these particles is increased by a contribution from ev and they are no longer BPS. How then can we understand the agreement of the BPS formula (3.7) on the string and four dimensional BPS formula on the Coulomb branch (2.5)? These elementary states of the vortex are to be thought of as four-dimensional elementary particles bound to the string, an interpretation which is clear from the brane picture of appendix A. In the center of the vortex string, one of the Higgs fields q_i vanishes and the theory effectively sits in a partial Coulomb phase.[8] The W-bosons and quarks which are carry charge under the corresponding U(1) may lower their mass to their Coulomb branch value by sitting where $q_i = 0$. For the $i^{\rm th}$ vortex, these are precisely the states with mass (3.7). The calculation above shows that these states actually re-obtain BPS status by this mechanism.

When the classical vortex theory has isolated vacua, it also admits topological kink solutions which contribute to the spectrum. Kinks in models of this type have been much studied in the literature, starting with Abraham and Townsend [9] and continued in [1, 26, 10]. The first order Bogomoln'yi equations describing the kink are given by,

$$\begin{aligned} \partial\sigma &= g^2\left(\sum_{i=1}^{N_c}|\psi_i|^2 - r\right) \\ \mathcal{D}\psi_i &= (\sigma - m_i)\psi_i \end{aligned}$$

where all derivatives are along the spatial worldvolume direction of the vortex string, and the fields are subject to the boundary conditions that they return to Vacuum i as $x \to -\infty$, and to Vacuum j as $x \to +\infty$. The BPS mass of such a kink is given by,

$$\mathcal{M}_{\rm kink} = r|m_i - m_j| = \frac{2\pi}{e^2}|m_i - m_j| \, . \tag{3.8}$$

Comparing with equation (2.2), we see that this coincides with the mass of the monopole with magnetic charge $h_a = \delta_{ai} - \delta_{aj}$, sitting at the root of the baryonic Higgs branch. In

[8] We thank M. Shifman and A. Yung for discussions and suggestions on this point.

fact, as shown in [5], the kink in the vortex string is precisely this magnetic monopole in the Higgs phase, with the string providing the flux line which whisks away the magnetic charge as required by Meissner. To see this, we examine the quantum numbers of the kink. As $x \to -\infty$, the vortex theory sits in the i^{th} vacuum state, corresponding to a magnetic flux in $\mathrm{U}(1)_i \subset \mathrm{U}(1)^{N_c}$. In the other direction, as $x \to +\infty$, the vortex sits in the j^{th} vacuum, the magnetic flux in the $\mathrm{U}(1)_j \subset \mathrm{U}(1)^{N_c}$ subgroup. Taking into account the direction of the flux, we see that the kink must provide a source of magnetic charge $h_a = \delta_{ai} - \delta_{aj}$, precisely that of the monopole. The magnetic flux assignment for a U(2) monopole is drawn in figure 1.

Finally, as with the monopoles of section 2, the kinks on the vortex string also admit a generalisation to dyons in which they are charged under the $\mathrm{U}(1)_D^{N_c-1}$ global flavour group of the vortex theory [9]. Such objects are known as q-kinks. Moreover, there is also an analog of the Witten effect [27] for these kinks so that, in the presence of a θ-angle, they pick up global electric charge [1].

To summarise, the classical BPS spectrum on the vortex string consists of a rich mix of both elementary and topological excitations. To write a central charge formula for the masses, we define the charge of a state under the $\mathrm{U}(1)_D^{N_c-1}$ global flavour symmetry to be S_i. We further define the topological charge T_i, such that a field configuration that tends toward Vacuum j as $x \to -\infty$ and to Vacuum k as $x \to +\infty$ has topological charge $T_i = \delta_{ij} - \delta_{jk}$. The masses of all BPS states are then given by $M = |Z|$ with the classical central charge given by,

$$Z = i \sum_{i=1}^{N_c} m_i (S_i + \tau T_i)$$

which agrees precisely with the classical central charge of the four-dimensional theory (2.6) if we equate the two-dimensional topological charge with the four-dimensional magnetic charge: $T_i = h_i$.

Now we turn to the description of the quantum spectrum of the vortex string. Once again exact results are available, although of a very different nature from the Seiberg-Witten curve that we employed in section 2. The trick, following Witten [28], is to integrate out the chiral superfields Ψ_i leaving an effective lagrangian for the vector multiplet fields. This is most elegantly expressed in terms of a twisted chiral superfield Σ whose lowest component is the complex scalar field σ, and includes F_{01} as part of the auxiliary field. In the presence of twisted masses, this calculation was first done in [3], resulting in the effective twisted superpotential,

$$\mathcal{W}(\Sigma) = \frac{i}{2}\tau\Sigma - \frac{1}{4\pi}\sum_{i=1}^{N_c}(\Sigma - m_i)\log\left(\frac{2}{\mu}(\Sigma - m_i)\right).$$

Assuming no singularities in the Kähler potential, the N_c quantum vacua of the theory are determined by the critical points of the twisted superpotential $\partial\mathcal{W}/\partial\Sigma = 0$ and are given by,

$$\prod_{i=1}^{N_c}(\sigma - m_i) - \Lambda^{N_c} \equiv \prod_{i=1}^{N_c}(\sigma - e_i) = 0$$

which we notice as the same equation describing the branch points of the Seiberg-Witten curve at the root of the baryonic Higgs branch (2.8). The classical BPS kinks which we described above also survive in this effective theory [29] although their mass is now corrected to include quantum effects. A kink interpolating between the Vacuum i and Vacuum j has mass $M_{\rm kink} = 2\Delta\mathcal{W} = 2\mathcal{W}(e_i) - 2\mathcal{W}(e_j)$. In the weak coupling regime $|m_i - m_j| \gg \Lambda$ the leading contribution is precisely the classical result (3.8). Deep in the strong coupling regime, $|m_i - m_j| \ll \Lambda$, quantum effects are dominant. The exact BPS mass of the kink can be captured by a correction to the central charge so that all BPS excitations of the string have masses $M = |Z|$, now with

$$Z = -i\sum_{i=1}^{N_c}(m_iS_i + m_{D\,i}T_i)$$

where all the quantum corrections are encoded in $m_{D,i}$, each a holomorphic function of m_j and Λ. Using the expressions above, we find that (up to an i-independent irrelevant constant)

$$m_{D\,i} = -2i\mathcal{W}(e_i) = \frac{1}{2\pi i}N_ce_i + \frac{1}{2\pi i}\sum_{j=1}^{N_c} m_j \log\left(\frac{e_i - m_j}{\Lambda}\right)$$

which we see coincides with the expression computed in four dimensions (2.10). Note that these two equations arose from very different origins: the degeneration of the Seiberg-Witten elliptic curve in four dimensions, and the critical points of the effective twisted superpotential in two dimensions. This agreement is the main result of [1].

Note that while we have shown, following [1], that the exact central charges agree in two and four dimensions, this does not necessarily imply that the spectra coincide. For this we have to show that the same quantum numbers S_i and T_i are realised in each theory. For example, from the perspective of the vortex string, we have seen that only kinks with quantum numbers $T_i = \delta_{ij} - \delta_{ik}$ are allowed classically. In contrast, in the four dimensional theory, there exist classical monopole configurations with arbitrary magnetic charge T_i, subject only to $\sum_i T_i = 0$. However, not all of these classical configurations may be realised as states in the quantum theory. It was shown in [1] that at weak coupling $|m_i - m_j| \gg \Lambda$, the allowed charges of quantum states do coincide between the two theories. Moreover, since the central charges agree, the curves of marginal stability where states may decay also coincide in the two theories. This strongly suggests that the spectra agree throughout the parameter space.

3.2 A weak coupling expansion

The results of the previous section reveal that the exact BPS mass spectrum of the vortex theory coincides with the exact BPS mass spectrum of the four-dimensional gauge theory. Powerful as these results are, it is constructive to examine them in the weak-coupling regime $|m_i - m_j| \gg \Lambda$. In this case, each holomorphic function $m_{D\,i}$ has the expansion,

$$m_{D\,i} = \frac{1}{2\pi i}\left(N_cm_i + \sum_{j=1}^{N_c}(m_j - m_i)\log\left(\frac{m_j - m_i}{\Lambda}\right) + \sum_{n=1}^{\infty} c_n(m_j)\Lambda^n\right)$$

where the log term arises as a one-loop contribution, while each term in the sum is due to a charge n instanton effect with an m_j dependent coefficient c_n. In the four-dimensional theory these are $\mathrm{U}(N_c)$ Yang-Mills instantons while, in the theory on the vortex string worldsheet, they are two-dimensional instantons which are usually referred to as semi-local vortices or $\mathbf{CP}^{N_c-1}$ lumps. In other words, from the perspective of the vortex string, Yang-Mills instantons look like semi-local vortices: a vortex within a vortex.[9] This is entirely analogous to the fact that, as we have seen above, a Yang-Mills monopole looks like a kink within a vortex. The fact that the coefficients c_n coincide term by term is presumably related to the observation of [4] that the moduli space of semi-local vortices is a submanifold of the moduli space of Yang-Mills instantons. It would be interesting to understand this agreement at the semi-classical level.

In fact, just as we derived the 1/4-BPS Bogomoln'yi equations for the monopole in the vortex [5], we may similarly derive the equations describing the Yang-Mills instanton in the Higgs phase in the presence of the vortex string. To do so, we set the hypermultiplet masses $m_i = 0$ to zero and work in four-dimensional euclidean space. We define a complex structure on $\mathbf{R}^4$ given by $z = x^2 + ix^3$ and $w = x^4 + ix^1$, and complete the four-dimensional action thus,

$$\begin{aligned}
\mathcal{L} &= \frac{1}{2e^2} F_{\mu\nu}F^{\mu\nu} + \sum_{i=1}^{N_f} |\mathcal{D}_\mu q_i|^2 + \frac{e^2}{2}\left(\sum_{i=1}^{N_f} q_i q_i^\dagger - v^2\right)^2 \\
&= \frac{1}{2e^2}\left(F_{12} - F_{34} - e^2\left(\sum q_i q_i^\dagger - v^2\right)\right)^2 + \sum \left(|\mathcal{D}_z q_i|^2 + \mathcal{D}_{\bar{w}} q_i|^2\right) + \\
&\quad + \frac{1}{2e^2}(F_{14} - F_{23})^2 + \frac{1}{2e^2}(F_{13} + F_{24})^2 + \frac{1}{e^2} F_{\mu\nu}F_{\rho\sigma}\epsilon^{\mu\nu\rho\sigma} + F_{12}v^2 - F_{34}v^2 \\
&\geq \frac{1}{e^2} F_{\mu\nu}F_{\rho\sigma}\epsilon^{\mu\nu\rho\sigma} + F_{12}v^2 - F_{34}v^2\,.
\end{aligned}$$

The terms left in the final line are all topological charges. We recognise the first as counting instanton number n when integrated over $\mathbf{R}^4$. The remaining two charges both count vortex strings. The term F_{12} is the topological charge for a string extended in the $x^3 - x^4$ plane as we have discussed above. The presence of the third charge F_{34}, which counts strings with worldvolume $x^1 - x^2$, reflects the fact that the most general solution to the Bogomoln'yi equations appear to contain more than we bargained for: orthogonal vortex strings, which share no worldvolume directions, together with Yang-Mills instantons. The Bogomoln'yi equations are

$$F_{12} - F_{34} = e^2\left(\sum_i q_i q_i^\dagger - v^2\right)\,, \quad F_{14} = F_{23}\,, \quad F_{13} = F_{24}\,, \quad \mathcal{D}_z q_i = 0\,, \quad \mathcal{D}_{\bar{w}} q_i = 0\,.$$

We see that these are an interesting mix of the usual self-dual Yang-Mills equations and the non-abelian vortex equations (3.1). As we mentioned, the most general solution seems likely to describe k_1 vortices with worldvolume in the $x^3 - x^4$ plane, another k_2 vortices

[9] From brane picture this is clear. In the IIA version described in appendix A, both objects arise as euclidean D0-branes lying in the D4 world-volume.

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with worldvolume in the $x^1 - x^2$ plane, and n Yang-Mills instantons. Such solutions likely preserve 1/8 supersymmetry. It would be interesting to study the properties of these solutions further. The relevance for the current work is restricted to the 1/4-BPS configurations with $k_2 = 0$.

4. $N_f > N_c$ and semi-local vortices

In this section we would like to generalise the story to $\mathcal{N} = 2$ U(N_c) supersymmetric QCD with $N_f > N_c$ massive hypermultiplets. In [2] a two-dimensional, non-compact sigma-model was presented whose mass spectrum coincides with that of this four-dimensional gauge theory. Here we confirm, using the results of [4], that this is indeed the theory living on the vortex string.

We start by taking generic masses for the hypermultiplets $m_i \neq m_j$ for $i \neq j$ where i and j now run from 1 to $N_f > N_c$. We also include a FI parameter v^2 from the beginning. The theory now has $N_f!/N_c!(N_f - N_c)!$ isolated vacua, labeled by the choice of N_c quarks which have an expectation value. Without loss of generality, we may choose the vacuum $\phi = \mathrm{diag}(m_1, \ldots, m_{N_c})$ with $\tilde{q}_i = 0$ and

$$q^a{}_i = \begin{cases} v\delta^a{}_i & a, i = 1, \ldots, N_c \\ 0 & \quad i = N_c + 1, \ldots, N_f \end{cases}$$

We will be interested in the BPS spectrum at this point on the Higgs branch. As in section 2, the W-bosons combine with N_c^2 quarks to form long multiplets. However, in contrast to the theory with $N_f = N_c$, there are now BPS quark states. These arise from the $(N_f - N_c)N_c$ quark hypermultiplet which parameterise the flat directions of the Higgs branch when $m_i = 0$. For non-vanishing m_i, these BPS quarks states have mass $M_{\mathrm{quark}} = |m_i - m_j|$ for $i = 1, \ldots, N_c$ and for $j = N_c + 1, \ldots, N_f$. The classical monopole spectrum remains much as in section 2. We can again compute the quantum corrections to the central charge using the Seiberg-Witten curve at the corresponding point on the Coulomb branch. The relevant formulae can be found in [2] so we shall be brief: the classical central charge at a generic point on the Coulomb branch is again given by (2.3). In our vacuum of choice $\phi = \mathrm{diag}(m_1, \ldots, m_{N_c})$ the degeneracies in the spectrum between quarks and W-bosons allows us to simplify the central charge as

$$Z = \sum_{i=1}^{N_c} m_i(S_i + m_{D\,i} h_i) + \sum_{i=N_c+1}^{N_f} m_i s_i \tag{4.1}$$

where the definitions are as in section 2, equation (2.6) and, classically, $m_{D\,i} = \tau$. Quantum mechanically, $m_{D\,i}$ can again be expressed as the integral of the Seiberg-Witten one-form λ_{SW} over a particular one-cycle of a new elliptic curve with N_c branch points e_i. Skipping the details, we simply quote the final result: $m_{D\,i}$ takes the form,

$$m_{D\,j} = \frac{1}{2\pi}\left[(2N_c - N_f)e_j - \sum_{i=1}^{N_c} m_i \log\left(\frac{e_j - m_i}{\Lambda}\right) + \sum_{i=N_c+1}^{N_f} m_i \log\left(\frac{e_j - m_i}{\Lambda}\right)\right]. \tag{4.2}$$

4.1 The semi-local vortex theory

We would now like to discuss the low-energy dynamics of the vortex string. Vortices in gauge theories with $N_f > N_c$ are known as "semi-local vortices", terminology which first arose in the abelian gauge theory with multiple Higgs fields [30]. As the gauge coupling e^2 is varied, these solitons interpolate between Nielsen-Olesen like vortices and sigma-model lumps on the Higgs branch of the theory. In non-abelian theories of the type considered here, they were studied in [4].

Semi-local vortices involve a subtlety not shared by those discussed in section 3: some of their zero modes are non-normalisable [31]. This means that the Manton metric on their moduli space includes some (logarithmically) divergent terms and these fluctuations are classically frozen. The non-normalisability of these modes also leads to subtleties in treating these objects quantum mechanically which, to our knowledge, have not been resolved in the literature.

In [4], a brane construction of both vortices and semi-local vortices was employed to extract the low-energy dynamics of the solitons. Although the resulting theory captured much information about vortex dynamics, it did not give the Manton metric on the moduli space. Indeed, for the case of semi-local vortices this discrepancy is most extreme since the brane construction provides a finite metric on the moduli space of semi-local vortices. Nevertheless, it was argued in [4] that as long as we restrict to BPS sectors of a supersymmetric gauge theory, then one should be able to use any of a class of metrics on the vortex moduli space since the questions reduce to calculating certain topological quantities. Here we present an example of this technique. Rather than using the non-normalisable metric on the semi-local vortex moduli space, we instead work with the simpler metric derived from the brane construction of [4]. The fact that we are able to reproduce the quantum spectrum of the four-dimensional gauge theory gives strong support in favour of this procedure. With this caveat in mind, we now describe the low-energy dynamics of the semi-local vortex [4]

Semi-local vortex theory: *$d = 1+1$, $\mathcal{N} = (2,2)$ supersymmetric* U(1) *with a single neutral chiral multiplet Z, N_c chiral multiplets Ψ_i of charge $+1$ and $(N_f - N_c)$ charged chiral multiplets $\tilde{\Psi}_m$ of charge -1. The Ψ_i have twisted masses m_i, $i = 1, \ldots, N_c$, while the $\tilde{\Psi}_m$ have twisted masses m_{N_c+m}, $m = 1, \ldots, N_f - N_c$.*

The FI parameter is given by $r = 2\pi/e^2$ as in section 2 and the D-term for the theory reads,

$$D = \sum_{i=1}^{N_c} |\psi_i|^2 - \sum_{m=1}^{N_f - N_c} |\tilde{\psi}_m|^2 - r\,.$$

After dividing by the U(1) gauge action, the equation $D = 0$ defines the Higgs branch of the theory for which, for $m_i = 0$, is isomorphic to the internal moduli space of semi-local vortices [4]. Note that, in contrast to the $\mathbf{CP}^{N_c-1}$ of section 2, the moduli space of semi-local vortices is non-compact. This reflects the fact that at large distances they look like sigma-model lumps, replete with a scaling modulus. When the masses are turned on $m_i \neq 0$, there are again only N_c isolated vacua in the theory, given by $|\psi_j|^2 = v^2\delta_{ij}$

and $\tilde{\psi}_m = 0$. Once again, these correspond to the N_c possible U(1) embeddings of the Nielsen-Olesen abelian vortex.

The semi-local vortex theory described above was previously studied in [2] where it was shown that the exact BPS spectrum indeed coincides with the spectrum of massive quarks and monopoles in the four-dimensional parent theory. Once again, we will be brief and make only a few choice comments. As in section 3, the anomalies in four and two dimensions are in agreement: for vanishing masses the $\mathrm{U}(1)_R$ symmetry is broken by instantons to $\mathbf{Z}_{2(2N_c-N_f)}$ in both cases. The one-loop logarithmic running of the FI parameter is given by,

$$r(\mu) = r_0 - \frac{2N_c - N_f}{2\pi}\log\left(\frac{M_{UV}}{\mu}\right) \tag{4.3}$$

which, agrees with the one-loop beta function of $e^2(\mu)$ in four-dimensions. Notice in particular that both two and four dimensional theories are asymptotically free for $N_f < 2N_c$ and infra-red free for $N_f > 2N_c$. Of particular interest is the critical case, $N_f = 2N_c$. On the four-dimensional Coulomb branch, with vanishing masses, the theory is conformal. Once we move onto the Higgs branch, the same is true of the theory of the vortex string. It may prove interesting to understand the relevance of this point.

Finally, the computation of the classical and quantum spectrum proceeds much as above — for full details see [2] — and reduces to computing the critical points of the effective superpotential,

$$\mathcal{W} = \frac{\tau\Sigma}{2} - \frac{1}{4\pi}\sum_{i=1}^{N_c}(\Sigma - m_i)\log\left(\frac{2}{\mu}(\Sigma - m_i)\right) + \frac{1}{4\pi}\sum_{m=N_c+1}^{N_f}(\Sigma - m_i)\log\left(\frac{2}{\mu}(\Sigma - m_i)\right).$$

The quantum corrected central charge takes the form (4.1), now with the $m_{D\,i} = -2i\mathcal{W}(e_i)$ where e_i are the N_c critical points of $\mathcal{W}$. Using the form of the superpotential above, we see that $m_{D\,i}$ is indeed given by (4.2). The exact BPS spectrum of the vortex string is in agreement with the BPS spectrum of its four-dimensional parent theory. Once again, the kinks have the interpretation of confined monopoles, while the elementary excitations of the string correspond to N_c^2 bound W-bosons, as well as $N_f(N_f - N_c)$ quark-string threshold states.

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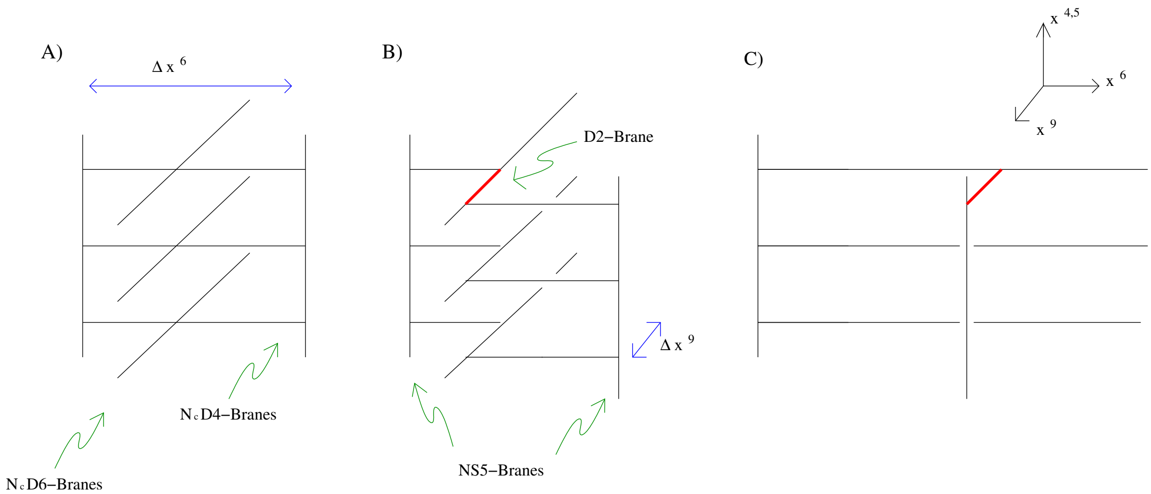

Figure 2: The type-IIA brane set-up. figure (a) shows the four dimensional theory at the root of the baryonic Higgs branch. In figure (b), the right-hand NS5-brane has slid in the x^9 direction and the gauge theory sits on the Higgs branch. We have included the D2-brane vortex string in red in this picture. In figure (c), we have moved the D6-branes off the page to the far-right, allowing us to read off the theory on the D2-brane.

A. The brane construction

In this section we review the brane derivation of the vortex theory given in [4] and present the (trivial) generalisation to include non-zero masses. While the construction of [4] was performed in the IIB string theory set-up of [32], resulting in vortices as particles in $d = 2 + 1$ dimensions, here we work with the T-dual IIA construction where the vortices appear as strings in a $d = 3 + 1$ dimensional gauge theory. Older relevant work can be found in [3, 33, 2]. Related brane constructions of vortices were recently discussed in [34].

Our brane configuration is drawn in figure 2. We use the well-known construction of $\mathcal{N} = 2$ theories in $d = 3 + 1$ dimensions realised on the worldvolume of N_c D4-branes suspended between two NS5-branes [20]. A further $N_f = N_c$ D6-branes give rise to hypermultiplets coming from $4 - 6$ strings. The spatial worldvolume directions of the branes are

$$\begin{aligned} NS5: &\ \ 12345 \\ D4: &\ \ 1236 \\ D6: &\ \ 123789 \\ D2: &\ \ 39\,. \end{aligned}$$

The gauge coupling e^2 and FI parameter v^2 are encoded in the separation Δx of the two NS5-branes,

$$\frac{1}{e^2} = \frac{\Delta x^6}{(2\pi)^2 g_s l_s}\,, \qquad v^2 = \frac{\Delta x^9}{(2\pi)^3 g_s l_s^3} \tag{A.1}$$

where g_s and $l_s = \sqrt{\alpha\prime}$ are the string coupling and string length respectively. The hypermultiplet masses and the vacuum expectation value of $\phi = \mathrm{diag}(\phi_1, \ldots, \phi_{N_c})$ are encoded in the x^4 and x^5 positions of the D-branes [20]

$$m_i = \left.\frac{x^4 + ix^5}{l_s^2}\right|_{D6_i}, \qquad \phi_i = \left.\frac{x^4 + ix^5}{l_s^2}\right|_{D4_i}. \tag{A.2}$$

In figure 2a we draw the brane configuration corresponding to the four dimensional theory with $v^2 = 0$ at the root of the baryonic Higgs branch $\phi = \mathrm{diag}(m_1, \dots, m_{N_c})$. In figure 2$b$, we have turned on the FI parameter v^2 by moving the right-hand NS5-brane out of the page in the x^9 direction. Here we also depict the vortex string, appearing as a D2-brane stretched the distance Δx^9 between the NS5-brane and the D3-brane.

To read off the vortex theory on the D2-brane, we first manipulate the branes a little. The field theory cares nothing for the x^6 position of the D6-branes and we may freely move them in this direction. Ther is one caveat however: they have non-zero linking number with the NS5-branes which ensures that D4-branes are created or destroyed if the two pass through each other [32]. We choose to move the D6-branes to the right. When they pass through the right-hand NS5-brane, the connecting D4-branes dissapear by flux conservation and the D6-branes are now attached only to the left-hand NS5-brane. After moving the D6-branes to $x^6 \to \infty$, the resulting configuration is shown in figure 2c. From this we may read off the gauge theory on the D2-brane as described in [3]. It is given by $d = 1+1$, $\mathcal{N} = (2,2)$ U(1) gauge theory. The gauge coupling constant g^2 and the FI parameter r are given by the separation of the NS5-branes,

$$\frac{1}{g^2} = \frac{\Delta x^9 l_s}{g_s}, \qquad r = \frac{\Delta x^6}{2\pi g_s l_s}.$$

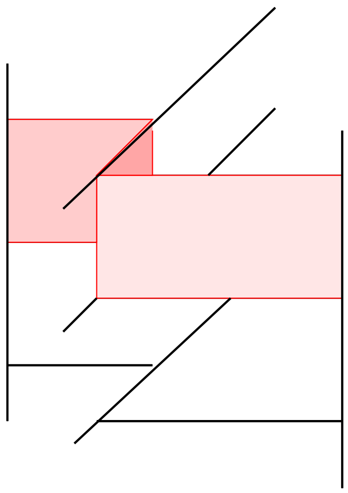

Figure 3: The confined monopole in the brane picture. The D2-brane worldsheet interpolates between the upper and middle D4-branes as $-\infty < x^3 < +\infty$.

As explained in [4], taking the decoupling limit of the four-dimensional gauge theory from the full string dynamics translates to the requirement that $g^2 \to \infty$. In contrast, r remains finite and, comparing with (A.1), is given by $r = 2\pi/e^2$ as promised. The matter content of the D2-brane theory includes a single free chiral multiplet, corresponding to motion in the $x^1 + ix^2$ direction, and N_c charged chiral multiplets arising from the $2-4$ strings. These chiral multiplets have a twisted mass given by the position of the D4-branes [3],

$$m_i = \left.\frac{x^4 + ix^5}{l_s^2}\right|_{D4_i}$$

which, for our choice of the baryonic Higgs branch, coincides with the hypermultiplet masses (A.2). This concludes the brane derivation of the vortex theory discussed in section 2.

From the brane picture, certain other aspects of the vortex dynamics become immediately obvious. In figure 2b, we drew the D2-brane attached to the upper D4-brane. This corresponds to a vortex string with magnetic flux in $B = \mathrm{diag}(1, 0, \dots, 0)$. It is clear from the brane picture that there exist a further $N_c - 1$ inequivalent vortex configurations in which the D2-brane is attatched to one of the other D4-branes. It is also simple to understand the confined monopole in this picture. We consider a D2-brane worldvolume which starts attatched to the upper D4-brane at $x^3 \to -\infty$, and then interpolates down to the middle D4-brane as $x^3 \to +\infty$. At intermediate steps, the D2-brane cannot simply be a line stretching distance Δx^9 as drawn in figure 2*b* since it has no where to end. The only possiblity is that the D2-brane bends in the x^6 direction to attatch itself to the NS5-branes. The final configuration is drawn in figure 3 and is similar to those considered in [3, 2].[10] Notice that as $v^2 \to 0$, and the separation Δx^9 of the NS5-branes vanishes, this stretched D2-brane indeed becomes the 't Hooft Polyakov monopole in the Coulomb phase.

B. Potential on the vortex moduli space

One of the key features of the vortex theory described in section 3 is the presence of twisted masses for the chiral multiplets. Here we provide a purely field theoretic derivation of this potential using a method developed in [35]. We then relate this potential to that arising on the Higgs branch of the vortex theory.

We are interested in non-abelian vortices in the four-dimensional theory described in section 2. If the hypermultiplet masses vanish $m_i = 0$, we may simply set the adjoint scalar $\phi = 0$ and study the vortex equations (3.1). The question we wish to answer here is how these solutions are lifted with the introduction of the masses m_i. To simplify matters, we take all m_i to be real, which allows us to restrict to real ϕ (the generalisation to complex masses is simple). Further, we will use the ability to shift ϕ to set $\sum_{i=1}^{N_f} m_i = 0$. A solution to the vortex equations (3.1) now has an extra contribution to its energy coming from the terms in the four dimensional action

$$V = \int d^2x \ \frac{2}{e^2} \,\mathrm{Tr}\, \mathcal{D}_z\phi \mathcal{D}_{\bar{z}}\phi + \sum_{i=1}^{N_f} q_i^\dagger \, (\phi - m_i)^2 q_i$$

which is to be evaluated on a particular configuration for the fields A_z and q_i solving (3.1). While q_i and A_z are fixed, ϕ may vary so as to minimize V. It satisfies,

$$\mathcal{D}^2\phi = e^2 \sum_{i=1}^{N_f} \,\{\phi, q_i q_i^\dagger\} - 2 q_i q_i^\dagger m_i \tag{B.1}$$

subject to the asymptotic condition $\phi \to \mathrm{diag}(m_1, \ldots, m_{N_c})$. In this appendix we show how to evaluate V for a given vortex solution.

The most general solution to the non-abelian vortex equations has $2kN_c$ parameters where k is the magnetic flux [4]. Let $\mathcal{V}_{k,N_c}$ denote the moduli space of solutions and choose coordinates X^p on $\mathcal{V}_{k,N_c}$ with $p = 1, \ldots, 2kN_c$. The tangent vectors of $\mathcal{V}_{k,N_c}$ are provided by the zero modes $(\delta_p A_z, \delta_p q_i)$ of the vortex which satisfy the linearised version of (3.1),

$$\mathcal{D}_z \, \delta_p A_{\bar{z}} - \mathcal{D}_{\bar{z}} \, \delta_p A_z \ = \ \frac{ie^2}{2} \sum_{i=1}^{N_f} \left(\delta_p q_i \, q_i^\dagger + q_i \, \delta_p q_i^\dagger \right)$$

[10] Similar brane configurations have been considered by J. Evslin in the context of [7].

$$\mathcal{D}_z\, \delta_p q_i \,=\, i\delta_p A_z\, q_i\,. \tag{B.2}$$

This is to be augmented by the gauge fixing condition arising from Gauss' law

$$\mathcal{D}_z\, \delta_p A_{\bar z} + \mathcal{D}_{\bar z}\, \delta_p A_z = -\frac{ie^2}{2}\sum_{i=1}^{N_f}\left(\delta_p q_i\, q_i^\dagger - q_i\, \delta_p q_i^\dagger\right) \tag{B.3}$$

The Manton metric on $\mathcal{V}_{k,N_c}$ is defined by the overlap of zero modes,

$$g_{pq} = \int d^2x\ \frac{2}{e^2}\,\mathrm{Tr}(\delta_{\{p}A_z)\,(\delta_{q\}}A_{\bar z}) + \sum_{i=1}^{N_f}(\delta_{\{p}q_i)\,(\delta_{q\}}q_i^\dagger) \tag{B.4}$$

Of particular interest will be the zero modes generated by symmetries, specifically the action of the $\mathrm{SU}(N)_{\mathrm{diag}}$ symmetry preserved by the vacuum when $m_i = 0$. As we have seen in section 3, for the case of a single vortex $k = 1$, this sweeps out the entire $\mathbf{CP}^{N_c-1}$ internal vortex moduli space [4, 6]. For higher k, it provides only a subset of the zero modes. In all cases, the action of the symmetry results in an $\mathrm{SU}(N_c)_{\mathrm{diag}}$ isometry of the moduli space metric g_{pq} with $N_c - 1$ mutually commuting holomorphic Killing vectors. These will be important in the following. As explained in section 2 of [4], the zero modes associated with this symmetry can be constructed uniquely from a given Lie algebra element $\Omega_0 \in su(N_c)_{\mathrm{diag}}$. The zero modes are given by,

$$\delta A_z = \mathcal{D}_z\Omega\,, \qquad \delta q = i(\Omega q - q\Omega_0) \tag{B.5}$$

where $\Omega = \Omega(z,\bar z)$, a function which, from (B.2) and (B.3), satisfies,

$$\mathcal{D}^2\Omega = e^2\sum_{i=1}^{N_f}\ \{\Omega, q_i q_i^\dagger\} - 2q_i q_i^\dagger\Omega_0 \tag{B.6}$$

subject to the boundary condition $\Omega(z,\bar z) \to \Omega_0$ as $|z| \to \infty$. Now let us choose a very special element Ω_0 which lies in the Cartan subalgebra of $su(N_c)_{\mathrm{diag}}$. We set

$$\Omega_0 = \mathrm{diag}(m_1, \ldots, m_{N_c})\,. \tag{B.7}$$

The crucial observation is that for this specific rotation, equation (B.6) coincides with the equation of motion for ϕ given in (B.1): we have $\Omega = \phi$. This allows us to write the excess energy of the vortices in terms of the overlap of these zero modes (B.5)

$$V = \int d^2x\ \frac{2}{e^2}\,\mathrm{Tr}\,\delta A_z\,\delta A_{\bar z} + \sum_{i=1}^{N_f}\delta q_i \delta q_i^\dagger\,. \tag{B.8}$$

We are now almost done. The final step is to decompose the specific rotation (B.7) into a basis of normalised rotations. It is somewhat simpler to work with the larger $u(N_c)_{\mathrm{diag}}$ Cartan sub-algebra and subsequently impose the vanishing trace condition on various objects. Let H_i denote the N_c mutually commuting generators and write $\Omega_0 = \sum_i m_i H_i$. We further denote by K_i the Killing vector on $\mathcal{V}_{k,N_c}$ which is generated by the action of H_i.

Note that, because of the traceless condition, these N_c Killing vectors are not all linearly independent but satisfy $\sum_i K_i = 0$. We may now express the zero mode (B.5) in this basis of tangent vectors,

$$\delta A_z = \left(\sum_{i=1}^{N_c} m_i K_i^p\right) \delta_p A_z \,, \qquad \delta q = \left(\sum_{i=1}^{N_c} m_i K_i^p\right) \delta_p q \,.$$

Finally, inserting this into (B.8) and using the defintion of the metric (B.4), we arrive at our promised result for excess vortex energy as a potential on $\mathcal{V}_{k,N_c}$ given by the length-squared of a particular Killing vector,

$$V = \sum_{i,j=1}^{N_c} (m_i K_i^p)\,(m_j K_j^q)\, g_{pq}\,. \tag{B.9}$$

B.1 From vortex theory to vortex moduli space

The vortex theory described in section 2 (and derived in appendix A using branes) is given in terms of a gauged linear sigma-model. The Higgs branch of the vortex theory coincides with the moduli space of vortices which, for a single vortex $k = 1$, is simply $\mathbf{C} \times \mathbf{CP}^{N_c-1}$. Here we would like to show how the potential (B.9) arises from the twisted mass terms in the vortex theory. The Higgs branch is defined by the D-term constraint

$$D = \sum_{i=1}^{N_c} |\psi_i|^2 - r = 0$$

modulo the U(1) gauge action which rotates each chiral multiplet equally: $\delta_{\rm gauge}\psi_i = i\psi_i$. The Higgs branch inherits a natural metric from the gauge theory through a mechanism known as the Kähler quotient. In the present context, this is simply the round Fubini-Study metric on $\mathbf{CP}^{N_c-1}$. The metric on the Higgs branch is defined in terms of a basis of tangent vectors $\delta_p\psi_i$, $p = 1, \dots, 2(N_c - 1)$ satisfying the linearised equations $\delta_p D = 0$ together with the gauge fixing constraint $\sum_i \psi_i^\dagger\, \delta_p\psi_i = 0$. The metric on the Higgs branch is then given by

$$g_{pq} = \sum_{i=1}^{N_c} (\delta_{\{p}\psi_i)\,(\delta_{q\}}\psi_i^\dagger)\,. \tag{B.10}$$

For the single $k = 1$ vortex considered here, all directions of the inernal moduli space are generated by the action of the SU$(N_c)_{\rm diag}$ symmetry.[11] We will match to the vortex moduli space calculation described above by following the action of this symmetry and, in particular, the $(N_c - 1)$ mututally commuting Killing vectors it generates on the Higgs branch. As above, it will prove useful to overcount and work with N_c Killing vectors subject to a constraint. Consider the N_c normalised zero modes arising from such the $su(N_c)_{\rm diag}$ action.

$$\delta_i\psi_j = i\psi_j\delta_{ij} - \frac{i}{r}|\psi_i|^2\psi_j\,. \tag{B.11}$$

[11] For the more general $k > 1$ theories discussed in [4], this statement is no longer true but the following methods can also be implemented.

These are not all linearly independent. If we denote the corresponding Killing vector on the Higgs branch as K_i, we have $\sum_{i=1}^{N_c} K_i = 0$. Note that the action has been normalised so that K_i coincides with the Killing vector on $\mathcal{V}_{1,N_c}$ defined above. It is simple to see how the masses m_i affect the Higgs branch. In the strict $g^2 \to \infty$ limit, they induce a potential given by the term from equation (3.4)

$$V = \sum_{i=1}^{N_f} |\psi_i|^2 (\sigma - m_i)^2 \tag{B.12}$$

where σ can vary so as to minimise V, giving rise to the solution

$$\sigma = \frac{1}{r} \sum_i m_i |\psi_i|^2 \,.$$

Substituting this into the potential (B.12), we see that we can express V purely in terms of geometrical objects on the Higgs branch: the metric (B.10) and the Killing vectors K_i arising from the action (B.11). We have,

$$\begin{aligned} V &= \sum_{i=1}^{N_c} |m_i|^2 |\psi_i|^2 - \frac{1}{r} |\sum_i m_i |\psi_i|^2|^2 \\ &= \sum_{i,j=1}^{N_c} (m_i K_i^p)\,(\bar{m}_j K_j^q)\, g_{pq} \end{aligned}$$

in agreement with the expression (B.9). It is heartwarming that the potentials derived from field theory and branes coincide.

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